The convergence of zero sequences in the indeterminate rational Stieltjes moment problem

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Joint work with
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Survey

- Crash course on Stieltjes moment problem
- The rational (multipoint) Stieltjes moment problem
- The convergence of zero sequences in the rational case

Problem setting

$$\text{Consider} \quad c_k = \int_0^\infty t^k d\mu(t), \quad k = 0, \pm 1, \pm 2, \ldots \tag{1}$$

! $\mathbf{c} = (c_k)_{k \in \mathbb{Z}}^\infty$
? all μ, pos, finite measure with $\mathrm{supp}(\mu)$ in $[0, \infty)$ satisfying (1).

$$\text{Stieltjes transform:} \quad S(z, \mu) = \int_0^\infty \frac{d\mu(t)}{z - t} \leftrightarrow \mu$$

Holomorphic in $\mathbb{C} \setminus \mathrm{supp}(\mu)$
Maps $\{z \in \mathbb{C} : \Im(z) > 0\} \to \{z \in \mathbb{C} : \Im(z) \leq 0\}$.

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Solution

- $\mathbf{c} \Rightarrow$ (moment) functional M defined on Λ (L-polynomials)
- $(M \ \& \ \mathbf{c} \succ 0) \Rightarrow$ SP $\langle f, g \rangle = M(fg)$
- $\langle \cdot, \cdot \rangle \Rightarrow$ OLPS $\varphi_n \in \Lambda_n \setminus \Lambda_{n-1}, \quad \varphi_n \perp \Lambda_{n-1}$
- OLPS $\Rightarrow$ 3TR

$$\begin{bmatrix} \sigma_{-1} & \sigma_0 \\ \varphi_{-1} & \varphi_0 \end{bmatrix} = \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \sigma_n \\ \varphi_n \end{bmatrix} = b_n \begin{bmatrix} \sigma_{n-1} \\ \varphi_{n-1} \end{bmatrix} + a_n \begin{bmatrix} \sigma_{n-2} \\ \varphi_{n-2} \end{bmatrix}$$

- 3TR $\Rightarrow$ CF: $\mathrm{K}_{n=0}^{\infty} \left(\frac{a_n}{b_n} \right) = (\Phi_n)_{n=0}^{\infty}, \quad \Phi_n = \frac{\sigma_n}{\varphi_n},$
  PFE$(\Phi_n) =$ PIQF$(\frac{1}{z-t}) \Rightarrow \Phi_n(z) = S(z, \mu_n)$
- $\Phi_{2m} \searrow \Phi_0 = S(\cdot, \mu_0)$ and $\Phi_{2m+1} \nearrow \Phi_\infty = S(\cdot, \mu_\infty)$
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Convergence

- zeros of φ_n
- $=$ poles of $\Phi_n = \sigma_n / \varphi_n$
- $=$ nodes of PIQF
- $=$ support μ_n
- $\mu_{2m} \to \mu_0, \quad \mu_{2m+1} \to \mu_\infty$
- $\Rightarrow \begin{cases} \text{zeros of } \varphi_{2m} \to \text{supp}(\mu_0) \\ \text{zeros of } \varphi_{2m+1} \to \text{supp}(\mu_\infty) \end{cases}$

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C. Bonan-Hamada, W.B. Jones, and O. Njåstad. *Zeros of orthogonal Laurent polynomials and solutions of the strong Stieltjes moment problem.* 2010 (submitted)

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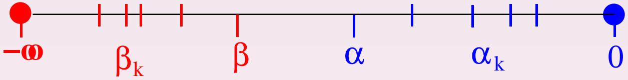

$$-\infty < \beta_k \le \beta < \alpha \le \alpha_k < 0$$

- $\Lambda_n \to \mathcal{L}_n = \text{span}\{\frac{p_n}{D_n} : p_n \in \Pi_n\}$
  $D_0 = 1,\ D_n = r_1 r_2 r_3 \cdots r_n$

$$r_{2m}(z) = (\beta_m - z), \qquad r_{2m+1}(z) = (\alpha_{m+1} - z).$$

- $\mathcal{L}_n$ alternatingly adds a pole at $(\alpha_m \approx 0)$ and at $(\beta_m \approx -\infty)$

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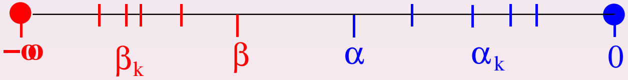

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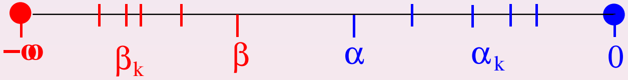

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Rational moments

- $\Lambda \cdot \Lambda = \Lambda$ BUT $\mathcal{L} \cdot \mathcal{L} \neq \mathcal{L}$ in general
- To define $\langle \cdot, \cdot \rangle$ one needs $M(fg)$, i.e., moments in $\mathcal{L} \cdot \mathcal{L}$

$$c_{i,j} = \int_0^\infty \Omega_i(t)\Omega_j(t)d\mu(t), \quad i,j = 0,\ldots,\infty, \quad \Omega_i = 1/D_i$$

Needs double index!

- $B = \{\beta_k : k = 1,2,\ldots\}$, $A = \{\alpha_k : k = 1,2,\ldots\}$
  $P = \mathrm{cl}(B) \cup \mathrm{cl}(A) \subset (-\infty, \beta] \cup [\alpha, 0]$ (Pole set)
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Indeterminate moment problem

- Recall $\Phi_n = \frac{\sigma_n}{\varphi_n}$ is nth cvg of $\mathrm{K}(\frac{a_n}{b_n})$

$$a_1 = \frac{W_1}{\alpha_1 - x}, \quad a_2 = \frac{W_2}{\beta_1 - z}, \qquad b_1 = \frac{Q_1 z + R_1}{\alpha_1 - z}, \quad b_2 = \frac{Q_2 + R_2(\alpha_1 - z)}{\beta_1 - z}$$

$$a_{2m} = \frac{W_{2m}(\beta_{m-1} - z)}{\beta_m - z}, \qquad b_{2m} = \frac{Q_{2m}(\beta_{m-1} - z) + R_{2m}(\alpha_m - z)}{\beta_m - z}, \quad m \geq 2$$

$$a_{2m+1} = \frac{W_{2m+1}(\alpha_m - z)}{\alpha_{m+1} - z}, \qquad b_{2m+1} = \frac{Q_{2m+1}(\alpha_m - z) + R_{2m+1}(\beta_m - z)}{\alpha_{m+1} - z}, \quad m \geq 1$$

$$Q_{2m} < 0, \quad Q_{2m+1} > 0, \quad R_{2m} > 0, \quad R_{2m+1} < 0, \quad W_n < 0$$

- MP is indeterminate iff

$$\sum \left| R_{2m} \frac{W_1 W_3 \cdots W_{2m-1}}{W_2 W_4 \cdots W_{2m}} \right| \leq \Gamma_e < \infty, \quad \sum \left| R_{2m+1} \frac{W_2 W_4 \cdots W_{2m}}{W_1 W_3 \cdots W_{2m+1}} \right| \leq \Gamma_o < \infty$$

$$\sum \left| Q_{2m} \frac{W_1 W_3 \cdots W_{2m-1}}{W_2 W_4 \cdots W_{2m}} \right| \leq \Lambda_e < \infty, \quad \sum \left| Q_{2m+1} \frac{W_2 W_4 \cdots W_{2m}}{W_1 W_3 \cdots W_{2m+1}} \right| \leq \Lambda_o < \infty$$

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\begin{aligned}
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&a_{2m} = \frac{W_{2m}(\beta_{m-1} - z)}{\beta_m - z}, && b_{2m} = \frac{Q_{2m}(\beta_{m-1} - z) + R_{2m}(\alpha_m - z)}{\beta_m - z}, \quad m \geq 2 \\
&a_{2m+1} = \frac{W_{2m+1}(\alpha_m - z)}{\alpha_{m+1} - z}, && b_{2m+1} = \frac{Q_{2m+1}(\alpha_m - z) + R_{2m+1}(\beta_m - z)}{\alpha_{m+1} - z}, \quad m \geq 1 \\
&Q_{2m} < 0, \quad Q_{2m+1} > 0, && R_{2m} > 0, \quad R_{2m+1} < 0, \quad W_n < 0
\end{aligned}
$$

- MP is indeterminate iff

$$
\sum \left| R_{2m} \frac{W_1 W_3 \cdots W_{2m-1}}{W_2 W_4 \cdots W_{2m}} \right| \leq \Gamma_e < \infty, \quad \sum \left| R_{2m+1} \frac{W_2 W_4 \cdots W_{2m}}{W_1 W_3 \cdots W_{2m+1}} \right| \leq \Gamma_o < \infty
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\sum \left| Q_{2m} \frac{W_1 W_3 \cdots W_{2m-1}}{W_2 W_4 \cdots W_{2m}} \right| \leq \Lambda_e < \infty, \quad \sum \left| Q_{2m+1} \frac{W_2 W_4 \cdots W_{2m}}{W_1 W_3 \cdots W_{2m+1}} \right| \leq \Lambda_o < \infty
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Zero sequences

- Recall $\frac{\sigma_{2m}}{\varphi_{2m}} \to \Phi_0 = S(\cdot, \mu_0)$, $\frac{\sigma_{2m+1}}{\varphi_{2m+1}} \to \Phi_\infty = S(\cdot, \mu_\infty)$
- Ordered zeros:
  for $\varphi_{2m+1} : Z_{2m+1} = \{x_k^{(2m+1)} : k = -m, \ldots, -1, 0, 1, \ldots, m\}$
  for $\varphi_{2m} \quad : Z_{2m} \quad = \{x_k^{(2m)} : k = -m, \ldots, -1, 1, \ldots, m\}$
  Zeros of φ_{2m+1} and φ_{2m} interlace.
- If we restrict ourselves to $(0, \infty)$ (exclude 0 and ∞)
  zeros $\Phi_\infty = \text{supp}(\mu_\infty) = \text{acc.pts. } \cup_{m=1}^{\infty} Z_{2m+1}$
  zeros $\Phi_0 = \text{supp}(\mu_0) = \text{acc.pts. } \cup_{m=1}^{\infty} Z_{2m}$
- For $x \in (\beta, \alpha)$ and μ any solution of the MP:[1]
  $$\Phi_\infty(x) = S(x, \mu_\infty) \le S(x, \mu) \le S(x, \mu_0) = \Phi_0(x)$$

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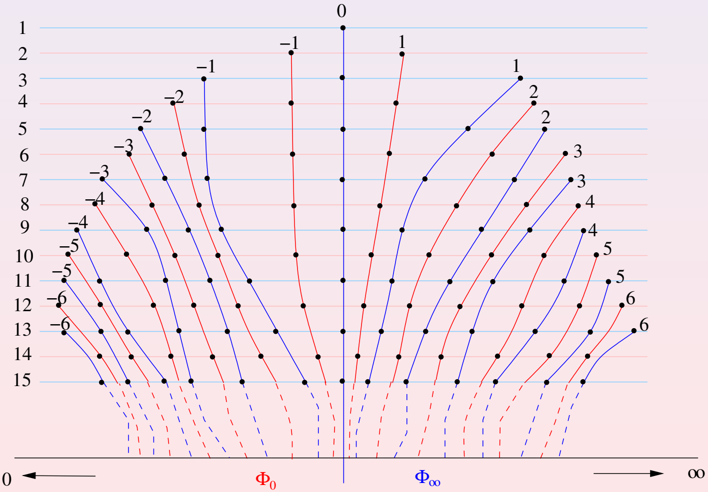
Convergence of zero sequences



Equivalent CF and its convergents

- Use an equivalence transformation for CF $\mathrm{K}(\frac{a_n}{b_n}) \to \mathrm{K}(\frac{1}{d_n})$.
  $d_{2m} = \frac{W_1 W_3 \cdots W_{2m-1} r_{2m}}{W_2 W_4 \cdots W_{2m} r_{2m-1}} b_{2m}, \; d_{2m+1} = \frac{W_2 W_4 \cdots W_{2m} r_{2m+1}}{W_1 W_3 \cdots W_{2m} r_{2m}} b_{2m+1}.$
- $d_n(x) > 0$ for $x \in (\beta, \alpha)$
- $\frac{\sigma_n}{\varphi_n} = \frac{\Sigma_n}{\Phi_n}$

$$\begin{bmatrix} \Sigma_{2m} \\ \Phi_{2m} \end{bmatrix} = \frac{\beta_m - z}{W_2 W_4 \ldots W_{2m}} \begin{bmatrix} \sigma_{2m} \\ \varphi_{2m} \end{bmatrix},$$
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Zeros

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- $\forall\mu$ solution of MP $S(x,\mu_\infty) \le S(x,\mu) \le S(x,\mu_0)$ on (β,α)
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